Example 2.1.8 *Consider $\mathbb{Z}$, the set of integers. Let a be a fixed integer. Let*

$$G = \{na \mid n \in \mathbb{Z}\}.$$

That is, G consists of all multiples of a. Note that $G \subseteq \mathbb{Z}$.

Now $0 = 0a \in G$. So it follows that G is nonempty. Because $+$ is commutative and associative on $\mathbb{Z}$ and G is a subset of $\mathbb{Z}$, it follows that $+$ is commutative and associative on G. Moreover, note that 0 is the identity element of G. Also for each $na \in G$, $-(na) = (-n)a \in G$ and

$$na + (-(na)) = 0 = (-(na)) + na.$$

We can now conclude that $(G, +)$ is a commutative group.

Let n be a fixed positive integer, Chapter 1 extensively describes the set $\mathbb{Z}_n$ and the binary relation $\equiv_n$on $\mathbb{Z}_n$. The next example shows that $\mathbb{Z}_n$ together with the binary relation $+_n$, as defined in that example, is a commutative group. The next two examples are, in fact, due to Gauss's, whose work yielded many new directions of research in Abelian groups.

Example 2.1.11 *Let*

$$\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}.$$

Note that $\mathbb{Q}[\sqrt{2}] \subseteq \mathbb{R}$. Now $0 = 0 + 0\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$. This shows that $\mathbb{Q}[\sqrt{2}] \neq \emptyset$. Define $+$ on $\mathbb{Q}[\sqrt{2}]$ as follows: for all $a + b\sqrt{2}, c + d\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$,

$$(a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + b) + (c + d)\sqrt{2}).$$

It is easy to see that $+$ is a binary operation on $\mathbb{Q}[\sqrt{2}]$. Note that $+$ is the usual addition. Because $\mathbb{Q}[\sqrt{2}] \subseteq \mathbb{R}$, it follows that $+$ is associative and commutative on $\mathbb{Q}[\sqrt{2}]$. Next, for all $a + b\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$,

$$(a + b\sqrt{2}) + (0 + 0\sqrt{2}) = a + b\sqrt{2} = (0 + 0\sqrt{2}) + (a + b\sqrt{2}).$$

Thus, $0 = 0+0\sqrt{2}$ is the identity element of $(\mathbb{Q}[\sqrt{2}], +)$. Note that the inverse of $a+b\sqrt{2}$ is $-a+(-b)\sqrt{2}$. Hence, $(\mathbb{Q}[\sqrt{2}], +)$ is a commutative groups.

In a similar manner, we can show that $(\mathbb{Q}[\sqrt{2}]\backslash\{0\}, \cdot)$ is a commutative group, where $\cdot$ is the usual multiplication. Note that the identity of $(\mathbb{Q}[\sqrt{2}]\backslash\{0\}, \cdot)$ is $1 = 1 + 0\sqrt{2}$ and the inverse of $a + b\sqrt{2} \neq 0$ is $\frac{a}{a^2-2b^2} - \frac{b}{a^2-2b^2}\sqrt{2}$.